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Quantization and Meaning of Observables Linear in Momentum

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A detailed study is made of the observables $\xi^i(x^k)p_i + f(x^k)$ linear in momentum on a Riemannian manifold: their quantization and (through quantum unitary transformations) physical meaning are discussed using geometrical methods.

1. INTRODUCTION

E. P. Wigner was known to go around asking the question, "What is the observable corresponding to $p+x$?" (Guenin, 1966). The problem is two-fold; the first part being the quantization of $p+x$ and the second the physical meaning of the quantized observable $Q(p+x)$ or in particular how one might measure $Q(p+x)$. This problem may be tackled by considering quantum canonical (unitary) transformations, a subject studied extensively in recent years by Moshinsky and co-workers (see Garcia-Calderon and Moshinsky, 1980, and references therein).

In this paper we shall study more generally the observable of coordinate form $\xi^i(x^k)p_i + f$ defined on a Riemannian configuration manifold, its quantization, and, via quantum unitary transformation, its physical meaning.

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2. QUANTIZATION

Let M be a complete Riemannian configuration manifold and T^*M its cotangent bundle; then an observable $P+f$ linear in momentum is a map $P+f$: $T^*M \to \mathbb{R}$ given by the prescription

$$\forall m \in T^*M, \qquad (P+f)(m) = P(m) + f(\pi(m)) \tag{1}$$

where P is a C^∞ momentum, f a C^∞ function on M, and π: $T^*M \to M$ the projector of the bundle T^*M. These observables generalize that considered by Wigner since, in every coordinate chart (x^i, p_i) of T^*M, $P+f$ assumes the tensor form $\xi^i p_i + f$. The procedure of quantization used[1] here consists of two simple steps:

1. Formal Quantization. To each observable $P+f$ we assign a unique symmetric operator in the natural Hilbert space $L^2(M)$ by the linear prescription

$$Q_0(P+f) = Q_0(P) + Q_0(f) \tag{2}$$

in which $Q_0(P) = -i\hbar(X + \frac{1}{2}\operatorname{div} X)$ and $Q_0(f) = f$ are defined on the set $C_0^\infty(M)$ of infinitely differentiable functions of compact support, and in which $X = \xi^i \partial/\partial x^i$ denotes the vector field associated with P (Wan and McFarlane, 1980).

Note that $Q_0(P+f)$ cannot, being symmetric but not self-adjoint, be identified with the quantum analog of $P+f$. Hence the following step.

2. Exact Quantization. We regard $P+f$ as quantizable if and only if $Q_0(P+f)$ is essentially self-adjoint, when the quantized observable $Q(P+f)$ is identified with the unique self-adjoint extension of $Q_0(P+f)$. Thus symbolically $Q(P+f) = Q_0^\dagger(P+f)$.

We here present a relationship between the quantizability conditions for a momentum P (Mackey, 1963; Wan and Viasminsky, 1977) and those of the corresponding observables $P+f$.

> *Theorem 1* (Appendix 1): *On the quantization of* $P+f$. $P+f$ is quantizable if and only if P is quantizable, i.e. if and only if P is complete,[2] when the corresponding quantum observable $Q(P+f)$

[1] A systematic discussion of quantization using the geometric methods of Kostant and Sourier will be given elsewhere.

[2] Strictly we require the completeness of almost every integral curve of the associated vector field X; we shall, however, continue to term this property "completeness" of the momentum P.

is given explicitly as

$$Q(P+f)=-i\hbar\left(D_X+\tfrac{1}{2}\operatorname{div}X+i\hbar^{-1}f\right) \tag{3}$$

on the domain

$$DQ(P+f)=\{\psi\in L^2(M)|\psi\in AC(X,M),Q(P+f)\psi\in L^2(M)\} \tag{4}$$

and hence satisfies the quantization rule

$$Q(P+f)=Q(P)+Q(f),\qquad Q(P)=Q_0^\dagger(P),\qquad Q(f)=Q_0^\dagger(f) \tag{5}$$

The symbol $AC(X, M)$ requires explanation: Let B be an open subset of M in which $X\neq 0$ and let B be sufficiently small that it is covered by a chart in which $X=\partial/\partial x^1$. $AC(X, M)$ is then the set of all functions η which are absolutely continuous with respect to x^1 (Fano, 1971; Shilov and Gurevich, 1966; Reed and Simon, 1972) in every such open set B. Equivalently in every such open set B, η may be expressed as the indefinite integral $\eta=\int^{x^1}\xi\,dx^1$ for some ξ and hence is differentiable with respect to X almost everywhere. This notation replaces that of previous papers (Wan and McFarlane, 1980, 1981) in which the notation $C^1(X, M)$ was adopted, which might have been construed as requiring differentiability everywhere.

3. UNITARY TRANSFORMATIONS AND PHYSICAL CONSIDERATIONS

The quantization scheme for a momentum P as given by Mackey (1963) (Wan and Viasminsky, 1977) is based upon a natural association between the complete vector field X generated by a momentum P, a one-parameter group of transformations of M, and a corresponding one-parameter group of unitary transformations of $L^2(M)$. Explicitly let σ_t, $t\in\mathbb{R}$, be the flow generated by X and let $\psi\in L^2(M)$; then the corresponding unitary transformations have the form

$$U^t_{Q(P)}\psi=\left(\sigma_t^*g^{1/2}/g^{1/2}\right)^{1/2}\sigma_t^*\psi \tag{6}$$

where g is the determinant of the Riemannian metric g_{ij} of M (a scalar density), and σ_t^* is a pull-back (Abraham and Marsden, 1978) of σ_t onto densities or functions as appropriate. $Q(P)$ is then defined to be the generator of the unitary group, so that $U_{Q(P)}^t = \exp[i\hbar^{-1}Q(P)t]$.

Now in the case of a quantizable observable $P+f$ we may construct a one-parameter group of unitary transformations of $L^2(M)$ by the prescription $U_{Q(P+f)}^t = \exp[i\hbar^{-1}Q(P+f)t]$ and relate these to the corresponding operators $U_{Q(P)}^t$ by means of the following theorems.

Theorem 2 (Appendix 1): *On the unitary operators* $U_{Q(P+f)}^t$. We have

$$U_{Q(P+f)}^t = e^{i\omega(f,t)/\hbar} U_{Q(P)}^t, \qquad \omega(f,t) = \int_0^t \sigma_{t'}^* f\, dt' \tag{7}$$

Theorem 3 (Appendix 2): *On the unitary equivalence of* $Q(P)$ *and* $Q(P+f)$. Suppose that the observable $P+f$ is quantizable and in addition that f admits a global solution to the equation $D_X F = f$; then,

$$U_{Q(P+f)}^t = e^{-iF/\hbar} U_{Q(P)}^t e^{iF/\hbar} \tag{8}$$

$$Q(P+f) = e^{-iF/\hbar} Q(P) e^{iF/\hbar} \tag{9}$$

and finally, denoting the spectral function of $Q(P)$, $Q(P+f)$ by $E_{Q(P)}(\lambda)$, $E_{Q(P+f)}(\lambda)$, respectively, we have

$$E_{Q(P+f)}(\lambda) = e^{-iF/\hbar} E_{Q(P)}(\lambda) e^{iF/\hbar} \tag{10}$$

Observe that while the equation $D_X F = f$ admits local solutions in general, it may not admit any global solutions. For Theorem 3 to be applicable the equation must admit as solution a global function on M. Note moreover that F may not be a C^∞ function on M. An example presented at the end of this section serves to illustrate the intricacy of the situation.

We may now return to discuss Wigner's question on the physical meaning of $p+x$. Let M be the Euclidean manifold $\mathbb{R}$ with a global Cartesian coordinate x conjugate to the momentum p: $p+x$ is clearly quantizable with quantum analog $Q(p+x) = -i\hbar d/dx + x$ on the domain given in equation (4). Since $D_{d/dx}F = x$ admits the global solution $F = \frac{1}{2}x^2$ we have by Theorem 3 that

$$Q(p+x) = VQ(P)V^\dagger, \qquad V = e^{-ix^2/2\hbar} \tag{11}$$

or that $Q(p+x)$ is unitarily equivalent to $Q(p)$, both possessing the same spectrum and having spectral functions related by

$$E_{Q(p+x)}(\lambda)=VE_{Q(P)}(\lambda)V^{\dagger} \tag{12}$$

This means that the expectation value of $Q(p+x)$ in any state ψ of its domain may be obtained from the expectation value of $Q(p)$ in the corresponding state $V^{\dagger}\psi$, so that concretely

$$\langle\psi|Q(p+x)|\psi\rangle=\langle V^{\dagger}\psi|Q(P)|V^{\dagger}\psi\rangle \tag{13}$$

this process not requiring the simultaneous measurement of the incompatible observables $Q(p)$ and $Q(x)$. In other words the measurement or determination of $Q(p+x)$ amounts to a measurement of $Q(p)$ together with a calculation by means of equation (12).

We can pursue this matter a little further by considering a simultaneous unitary transformation of $Q(x)$ and $Q(p)$ to $VQ(x)V^{\dagger}$ and $VQ(p)V^{\dagger}=Q(p+x)$. We then see that $Q(p+x)$ and $Q(x)$ are in fact equally good canonical variables in which $Q(p+x)$ is now the momentum conjugate to $Q(x)$. The nonuniqueness of a momentum conjugate to a given coordinate should not be regarded as puzzling since a similar ambiguity exists in classical mechanics: our theorems here serve to deal more systematically with this problem.

Finally it should be observed that while Theorem 2 applies generally Theorem 3 operates under a restriction on F or as is equivalent on f, and that in general $Q(P+f)$ and $Q(P)$ are not unitarily similar. This may be illustrated by means of the usual angular momentum observable L_z on $\mathbb{R}^2$: The spectrum of $Q(L_z)$ is wholly discrete with eigenfunctions $\varphi_n(r,\theta)=A_n(r)\exp(in\theta)$, and eigenvalues $n\hbar$, $n\in\{0,\pm1,\pm2,\ldots\}$. Here (r,θ) denote polar coordinates in $\mathbb{R}^2$ and $A_n(r)$ is such as to normalize φ_n. Now consider the observable $Q(L_z+f)$, $f\in C^{\infty}(\mathbb{R}^2)$ which results in the following formal eigenfunctions:

$$\psi_\mu(r,\theta)=A_\mu(r)e^{i\mu\theta}\exp\left[-i\hbar^{-1}\int_0^{\theta}f(r,\theta')\,d\theta'\right] \tag{14}$$

The functions $\psi_\mu(r,\theta)$ become proper, rather than formal, eigenfunctions of $Q(L_z+f)$ if the single-valuedness condition $\psi_\mu(r,\theta)=$

$\psi_\mu(r, \theta+2\pi)$ is satisfied. This imposes a restriction on f, namely, that the θ-mean $\langle f\rangle_\theta = (2\pi)^{-1}\int_0^{2\pi} f(r,\theta')\,d\theta'$ must be a numerical constant independent of r. This being satisfied we then have $\mu_n = n + \hbar^{-1}\langle f\rangle_\theta$, $n \in \{0, \pm 1, \pm 2, \ldots\}$, so that the spectrum of $Q(L_z + f)$ is wholly discrete with eigenvalues $\hbar\mu_n$. Otherwise $Q(L_z + f)$ has no proper eigenvalues and the spectrum is wholly continuous showing that $Q(L_z + f)$ is nontrivially distinct from $Q(L_z)$. We now demonstrate that coincidence of the spectra of $Q(L_z)$ and $Q(L_z + f)$ is assured by the conditions of Theorem 3, this coincidence occurring only when $\langle f\rangle_\theta = 0$. To this end observe that the equation $D_{L_z}F = f$ admits local solutions of the form $\int_0^\theta f(r,\theta')\,d\theta'$ + const. Such a solution is a global function on M if and only if it is single valued, iff $\int_0^{2\pi} f(r,\theta')\,d\theta' = 0$. In this case Theorem 3 operates clearly exhibiting why $Q(L_z)$ and $Q(L_z + f)$ have identical spectra.

4. QUANTUM GLOBAL MEASURABILITY

The concept of the quantum global measurability of a momentum, based upon the existence of certain global bounds to the errors of locally conducted measurements, and recently introduced by Wan and McFarlane (1980, 1981), is readily generalized to the observables $P + f$ linear in momentum (or at least to those satisfying the premises of Theorem 3) for the simple reason that unitary transformations preserve the scalar product. More explicitly we have that, upon setting $\Psi = \psi \exp(iF/\hbar)$,

$$\Delta Q_0(P+f)|_\Psi = \|(Q_0(P+f) - \langle\Psi|Q_0(P+f)|\Psi\rangle)\Psi\| = \Delta Q_0(P)|_\psi \tag{15}$$

Hence, upon noting that ψ and Ψ have the same support, we may deduce that, for any sequences ψ_n of wave functions in the domain of $Q_0(P)$, the corresponding sequences of uncertainties $\Delta Q_0(P+f)|_{\Psi_n}$ and $\Delta Q_0(P)|_{\psi_n}$ converge or diverge together. Consequently, $P + f$ is quantum globally measurable if and only if so also is P, and therefore the properties associated with quantum global measurability (Wan and McFarlane, 1980, 1981) apply equally to the observables $P + f$. In particular the quantum global measurability of an observable $P + f$ assures its quantizability (in the sense of Theorem 1).

5. LOCAL OBSERVABLES

Recently we have introduced the notion of formal local quantum observable[3] $Q_0^A(P)$ associated with an open subset A of M and generated by a global momentum P, and then discussed the conditions necessary that $Q_0^A(P)$ be essentially self-adjoint and therefore correspond to a local quantum observable $Q^A(P)$ (McFarlane and Wan, 1981). Explicitly for any open subset A of M introduce a projector onto A by

$$\pi\psi(x)=\begin{cases}\psi(x), & x\in A,\\ 0, & x\notin A\end{cases} \qquad \forall\psi\in L^2(M), \tag{16}$$

then the formal local observable generated by the momentum P is defined to be $Q_0^A(P)=\pi Q_0(P)\pi$. We may now generalize the concept of local observable to objects of the form $P+f$ (or at least those satisfying the conditions of Theorem 3) by defining the formal local observables associated with $P+f$ as simply

$$Q_0^A(P+f)=\pi Q_0^A(P+f)\pi=Q_0^A(P)+Q_0^A(f) \tag{17}$$

a definition which leads immediately to the following two theorems:

Theorem 4 (Appendix 3): *On the existence of local observables* $Q^A(P+f)$. The formal quantum observable $Q_0^A(P+f)$ is essentially self-adjoint if and only if so also is $Q_0^A(P)$, when the corresponding self-adjoint extension of $Q_0^A(P+f)$ assumes the form

$$Q^A(P+f)=Q^A(P)+Q^A(f), \qquad Q^A(f)=\pi Q(f)\pi \tag{18}$$

Theorem 5 (Appendix 3): *On the reconstruction of* $Q(P+f)$. Let A_α be a family of open subsets forming a partition of M except possibly for a set of measure zero. Then for a quantizable $P+f$ we have

$$Q(P+f)=\sum_\alpha Q^{A_\alpha}(P+f) \text{ iff } Q(P)=\sum_\alpha Q^{A_\alpha}(P) \tag{19}$$

For the somewhat intricate conditions under which the observables $Q^A(P)$ and the expansion $\Sigma_\alpha Q^{A_\alpha}(P)$ exist, the reader is referred to

[3]By a formal quantum observable is meant a symmetric operator which may or may not be essentially self-adjoint.

McFarlane and Wan, 1981. We note in conclusion that Theorems 4 and 5 are valid even when f does not satisfy the premise of Theorem 3.

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APPENDIX 1. QUANTIZATION AND UNITARY TRANSFORMATIONS

Lemma A1.1: On the function $\omega(f,t)$ of Theorem 2. (i) For any fixed $m \in M$, $\omega(f,t)$ is a function of t satisfying $\partial\omega(f,t)/\partial t = \sigma_t^* f$. (ii) For any fixed $t \in \mathbb{R}$, $\omega(f,t)$ is a function on M satisfying

$$\text{(a)} \quad \sigma_s^*\omega(f,t) = \omega(f,t+s) - \omega(f,s)$$

$$\text{(b)} \quad \omega(f,t) \in C^\infty(X,M), \qquad D_X^n\omega(f,t) = (\sigma_t^* - 1)D_X^{n-1}f$$

in which $C^\infty(X,M)$ is the set of functions infinitely differentiable with respect to X.

Proof. (i) Trivial. (ii) (a) $\sigma_s^*\omega(f,t) = \sigma_s^*\int_0^t \sigma_{t'}^* f\,dt' = \int_0^t \sigma_s^*(\sigma_{t'}^* f)\,dt'$ (Greub, et al.[4]). Hence

$$\sigma_s^*\omega(f,t) = \int_0^t \sigma_{s+t'}^* f\,dt' = \int_0^{t+s} \sigma_{t'}^* f\,dt' - \int_0^s \sigma_{t'}^* f\,dt' = \omega(f,t+s) - \omega(f,s)$$

$$\begin{aligned}\text{(b)} \quad D_X\omega(f,t) &= \lim_{s\to 0}\left[\sigma_s^*\omega(f,t) - \omega(f,t)\right]s^{-1} \\ &= \lim_{s\to 0}\left[\omega(f,t+s) - \omega(f,s) - \omega(f,t)\right]s^{-1} \\ &= \lim_{s\to 0}\left[(\sigma_t^* - 1)\omega(f,s)\right]s^{-1} = (\sigma_t^* - 1)f\end{aligned}$$

Moreover, since D_X commutes with σ_t^* (Loomis and Sternberg, 1968), we have

$$(\sigma_t^* - 1)D_X^{n-1}f = D_X^{n-1}(\sigma_t^* - 1)f = D_X^n\omega(f,t)$$

as was required. ∎

[4]We wish to thank A. Cant for pointing out this reference.

Proof of Theorem 2. (i) $U^t_{Q(P+f)} = \exp[i\omega(f,t)/\hbar]U^t_{Q(P)}$ is obviously unitary.

$$
\begin{aligned}
\text{(ii)}\quad U^t_{Q(P+f)} \circ U^s_{Q(P+f)} &= \left(\sigma_t^* g^{1/2}/g^{1/2}\right)^{1/2} \exp[i\omega(f,t)/\hbar] \\
&\quad \times \sigma_t^*\left\{\left(\sigma_s^* g^{1/2}/g^{1/2}\right)^{1/2} \exp[i\omega(f,s)/\hbar]\sigma_s^*\right\} \\
&= \left(\sigma_t^* g^{1/2}/g^{1/2}\right)^{1/2} \exp[i\omega(f,t)/\hbar] \\
&\quad \times \left(\sigma_{t+s}^* g^{1/2}/\sigma_t^* g^{1/2}\right)^{1/2} \exp[i\sigma_t^*\omega(f,s)/\hbar]\sigma_{s+t}^* \\
&= \left(\sigma_{t+s}^* g^{1/2}/g^{1/2}\right)^{1/2} \exp[i\omega(f,t)/\hbar] \\
&\quad \times \exp\{i[\omega(f,t+s)-\omega(f,t)]/\hbar\}\sigma_{s+t}^* \\
&= U^{t+s}_{Q(P+f)}
\end{aligned}
$$

Hence $U^t_{Q(P+f)}$ forms a one-parameter group of unitary operators in $L^2(M)$ which is readily seen to be weakly continuous. The existence of the self-adjoint generator $Q(P+f)$ satisfying $U^t_{Q(P+f)} = \exp[i\hbar^{-1}Q(P+f)t]$ is now assured by Stone's theorem (Prugovecki, 1971). (iii) It remains only to exhibit the form of the generator $Q(P+f)$, which we accomplish by a series of lemmas:

> *Lemma A1.2: On a symmetric restriction of $Q(P+f)$.* The operator $Q(P+f)$ when restricted to $C_0^\infty(M)$ has the form
>
> $$Q_0(P+f) = -i\hbar\left(X + \tfrac{1}{2}\operatorname{div} X + i\hbar^{-1} f\right) = Q_0(P) + Q_0(f)$$

Proof. By direct calculation:

$$\forall \psi \in C_0^\infty(M), \qquad Q_0(P+f)\psi = -i\hbar \left.\frac{dU^t_{Q(P+f)}}{dt}\psi\right|_{t=0} \qquad \blacksquare$$

> *Lemma A1.3: On the essential self-adjointness of $Q_0(P+f)$.* The operator $Q_0(P+f)$ of Theorem 1 is essentially self-adjoint if P is complete almost everywhere.

Proof. If P is complete almost everywhere, then $U^t_{Q(P+f)}$ forms a group generating a self-adjoint operator $Q(P+f)$. Now $Q_0(P+f)$ is a symmetric restriction of $Q(P+f)$ by Lemma A1.2. Moreover it is clear that the domain of $Q_0(P+f)$ that is $C_0^\infty(M)$ is invariant under the action of $U^t_{Q(P+f)}$

so that by Nelson's lemma (Abraham and Marsden, 1978) $Q_0(P+f)$ is essentially self-adjoint. ■

Lemma A1.4: On the adjoint of $Q_0(P+f)$. As described explicitly in (3), (4), and (5),

$$Q_0^\dagger(P+f)=Q(P)+Q(f)$$

Proof. The expression for $Q(P)$ and its proof are obtained by Wan and McFarlane (1980). A slight modification of this proof to take account of f yields the result. Note that this lemma holds whether P is complete or not. ■

Lemma A1.5: On the completeness of P. If the operator $Q_0(P+f)$ of Theorem 1 with a bounded function f is essentially self-adjoint, then P is complete almost everywhere.

Proof. If f is bounded, then $Q_0^{\dagger\dagger}(P+f)=Q_0^\dagger(P+f)$ implies $Q^\dagger(P)+Q(f)=Q(P)+Q(f)$ (Riesz and Sz.-Nagy, 1955). This means $Q^\dagger(P)=Q(P)$, i.e., $Q_0(P)$ is essentially self-adjoint, hence the completeness of P. This proof fails when f is an unbounded function. ■

Lemma A1.6: On the completeness of P.[5] If the operator $Q_0(P+f)$ of Theorem 1 is essentially self-adjoint, then P is complete almost everywhere.

Proof. Let A be a simply connected open subset with a compact closure in M such that on every point of A the vector field X has an incomplete maximal integral curve, i.e., the local flow σ_t on A fails to be defined for, say, $t \geqslant \Gamma > 0$. Generalizing the method of Nelson (Abraham and Marsden, 1978) we introduce the function

$$\eta(P,f)=\int_{-\infty}^{+\infty} dt\, e^{-t} e^{i\omega(f,t)/\hbar}\left(\sigma_t^* g^{1/2}/g^{1/2}\right)^{1/2}\sigma_t^*\chi_A$$

where σ_t^* acting on a function or a density is taken to be zero whenever it is undefined, and χ_A is the characteristic function of A. Observe that $\sigma_t^*\chi_A$, hence the above integrand and η, is zero at $m \in M$ if $\sigma_t(m)$ is not in A. It can be readily verified that $\eta(P,f)$ is an element of $L^2(M)$ and that for any $\varphi \in C_0^\infty(M)$, we have $\langle \eta(P,f) | (Q_0^\dagger(P+f)-i\hbar)\varphi\rangle = 0$. If $\|\eta(P,f)\| \neq 0$ then $Q_0(P+f)$ is not essentially self-adjoint (Hellwig, 1964). Now let us

[5] We wish to thank B. Angermann for pointing out some errors in a previous version of the proof.

confine our attention in what follows to points $m \in M$ such that $\sigma_m(t) \in A$ for some t. For any such point m let $\Delta T_m = (T_1(m), T_2(m)) = \{t \colon \sigma_t(m) \in A\}$, then $\eta(P, f)$ at m is obtained by integrating over ΔT_m only. Now for $t \in \Delta T_m$ we have $\omega(f, t) = \omega_1(f) + \omega_2(f, t)$, where

$$\omega_1(f) = \int_0^{T_1(m)} \sigma_s^* f(m)\, ds, \qquad \omega_2(f, t) = \int_{T_1(m)}^{t} \sigma_s^* f(m)\, ds$$

and

$$\eta(P, f) = e^{i\omega_1(f)/\hbar} \xi(P, f)$$

where

$$\xi(P, f) = \int_{\Delta Tm} dt\, e^{-t} e^{i\omega_2(f,t)/\hbar} \left(\sigma_t^* g^{1/2} / g^{1/2} \right)^{1/2} \sigma_t^* \chi_A$$

Observe that $\omega_2(f, t)$ is determined entirely by the values of f in A. Let f_A be a bounded C^∞ function which coincides with f in A. Let $f_0 = f - f_A$; then $\omega_2(f_0, t) = 0$ for $t \in \Delta T_m$ and $\|\eta(P, f_0)\| = \|\xi(P, f_0)\| \neq 0$ since the integral for $\xi(P, f_0)$ has a real and positive integrand. This means that $Q_0(P + f_0)$ is not essentially self-adjoint, a result which, bearing in mind the boundedness of f_A, implies that $Q_0(P + f) = Q_0(P + f_0) + Q_0(f_A)$ is not essentially self-adjoint. ■

APPENDIX 2: ON THE UNITARY EQUIVALENCE OF $Q(P + f)$ AND $Q(P)$

Proof of Theorem 3. Let F be a function on M satisfying the equation $D_X F = f$ everywhere; then

$$\omega(f, t) = \int_0^t \sigma_{t'}^* f\, dt' = \int_0^t D_X(\sigma_{t'}^* F)\, dt'$$

$$= D_X \int_0^t \sigma_{t'}^* F\, dt' = D_X \omega(F, t) = (\sigma_t^* - 1) F$$

Hence by direct calculation

$$U^t_{Q(P+f)} = e^{i(\sigma_t^* - 1)F/\hbar} U^t_{Q(P)} = e^{-iF/\hbar} U^t_{Q(P)} e^{iF/\hbar}$$

from which the remaining results of the theorem can readily be established. ■

APPENDIX 3: ON LOCAL OBSERVABLES

Proof of Theorem 4. $Q_0^A(P+f)=Q_0^A(P)+\pi Q_0(f)\pi$ implies that

$$\left(Q_0^A(P+f)\right)^\dagger=\left(Q_0^A(P)\right)^\dagger+\pi Q_0^\dagger(f)\pi,$$

$$\left(Q_0^A(P+f)\right)^{\dagger\dagger}=\left(Q_0^A(P)\right)^{\dagger\dagger}+\pi Q_0^{\dagger\dagger}(f)\pi$$

since π and $\pi Q_0(f)\pi$ are bounded operators (Riesz and Sz.-Nagy, 1955). Moreover, $Q_0^\dagger(f)=Q_0^{\dagger\dagger}(f)=Q(f)$, so that $(Q_0^A(P+f))^\dagger=(Q_0^A(P+f))^{\dagger\dagger}$ if and only if $(Q_0^A(P))^\dagger(Q_0^A(P))^{\dagger\dagger}$ as required. ■

Proof of Theorem 5. $Q^{A_\alpha}(P+f)=Q^{A_\alpha}(P)+Q^{A_\alpha}(f)$ implies that $\Sigma_\alpha Q^{A_\alpha}(P+f)=\Sigma_\alpha Q^{A_\alpha}(P)+\Sigma_\alpha Q^{A_\alpha}(f)$. It then follows from $Q(P+f)=Q(P)+Q(f)$ that $Q(P+f)=\Sigma_\alpha Q^{A_\alpha}(P+f)$ if and only if $Q(P)=\Sigma_\alpha Q^{A_\alpha}(P)$. ■

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